Heisenberg Uncertainty Principle and the Quantum Harmonic Oscillator

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Abstract

Further to our discussions today in class, where have managed to confuse everyone and myself with question 9 (so apologies for that!) I have now had a think and prepared this document with some derivations which should clear things up. We will address:

1. Do the excited states have also have minimum uncertainty?
2. and what physical interpretation can we make by making use of the correspondence principle?

1 General Derivations of the Root Mean Square Deviations of $\hat{x}$ and $\hat{p}$

First we recall that $\hat{x}$ and $\hat{p}$ can be written in terms of the creation and annihilation operators:

$$x = \frac{1}{\sqrt{2k}}(\hat{a}^{\dagger} + \hat{a}) \tag{1}$$

$$p = i\sqrt{\frac{m}{2}}(\hat{a}^{\dagger} - \hat{a}) \tag{2}$$

Instead of deriving using the ground state as suggested by the question, let us use a general stationary state $|n\rangle$ where $n \in \mathbb{Z}^{+}$. The root mean square (RMS) deviation is defined as:

$$\Delta\Omega = \sqrt{\langle\Omega^2\rangle - \langle\Omega\rangle^2} \tag{3}$$

for some general observable Ω. Beginning with Δx:

$$\Delta x = \left(\langle n| x^2 |n\rangle - \langle n| x |n\rangle^2 \right)^{\frac{1}{2}}. \tag{4}$$

Which would expand to give:

$$\Delta x = \left[\langle n| \left(\left(\frac{1}{\sqrt{2k}} \left(a^\dagger + a \right) \right) \right)^2 |n\rangle - \underbrace{\left(\langle n| \frac{1}{\sqrt{2k}} \left(a^\dagger + a \right) |n\rangle \right)^2}_{0} \right]^{\frac{1}{2}} \tag{5}$$

where we have identified the second term to be 0. Because of orthogonality. (This also makes physical sense as the mean position of an oscillator should be 0 in a symmetric potential.) The first term, when expanded, will give:

$$\Delta x = \left[\frac{1}{2k} \underbrace{\langle n| a^\dagger a^\dagger |n\rangle}_{0} + \underbrace{\langle n| aa |n\rangle}_{0} + \underbrace{\langle n| a^\dagger a |n\rangle}_{\neq 0} + \underbrace{\langle n| aa^\dagger |n\rangle}_{\neq 0} \right]^{\frac{1}{2}} \tag{6}$$

where we note that only the last two terms are non-zero. The terms with a^2 and $a^{\dagger^2}$ do not contribute to the matrix elements because both will give a state orthogonal to the bra state on the left. The two surviving terms will give:

$$\langle n| (a^\dagger a + aa^\dagger) |n\rangle = 2n + 1 \tag{7}$$

so consequently:

$$\Delta x = \sqrt{\left(n + \frac{1}{2} \right) \frac{\hbar}{m\omega}} \tag{8}$$

In a similar way, the deviation in p is:

$$\Delta p = \sqrt{\langle n|p^2|n\rangle} = \sqrt{\left(n + \frac{1}{2} \right) m\hbar\omega} \tag{9}$$

so their products are:

$$\Delta x \Delta p = \left(n + \frac{1}{2} \right) \hbar. \tag{10}$$

We can rewrite Eqn 10 as a lower bound, as $n \geq 0$ always:

$$\boxed{\Delta x \Delta p \geq \frac{\hbar}{2}} \tag{11}$$

which is the Heisenberg Uncertainty Principle. **Question 9 asks**:*do excited states also have minimum uncertainty?* The correct answer according to Eqn 10 would be **no**. As the uncertainty increases with n. This is an important fact that comes about due to the form of our ground state wavefunction. When $n = 0$, our wavefunction ψ_0 has the form of a Gaussian wavepacket which minimises the uncertainty. Perhaps that is the most important take-home message. **The ground state minimises the position-momentum uncertainty.**

2 Comparison to Classical Oscillators

There is really no direct analogue of the quantum harmonic oscillators. Because in quantum mechanics, the solutions are what's called stationary states, a concept that does not exist classically. However, we can construct a classical probability density by assuming our oscillator is governed by the equation $x(t) = q_0 \sin(\omega t)$ where q_0 is some constant and ω is a fixed angular frequency. I won't give the exact form here as it's not relevant. But should you be interested, consult Shankar's *Principles of Quantum Mechanics* chapter 9. By plotting the two probabilities, we can clearly see correspondence being demonstrated. This is especially true for high-lying states ($n \to \infty$)